Coupled Translational and Rotational Motions of a Sphere in a Fluid

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(Z. Naturforsch. **23 a**, 1095—1101 [1968]; received 8 April 1968)

Starting from the exact steady state Navier-Stokes equation and taking the quadratic inertia term into account in an appropriate way one can show the existence of a coupling of the translational and rotational motions of a rigid sphere in an incompressible fluid. No-slip boundary conditions are used. The coupling consists in a transverse force proportional to the cross product of the linear and angular velocities of the sphere. This force is closely related to the Magnus effect. It is of interest for the theory of Brownian motion of particles with internal degrees of freedom.

The coupling of translational and rotational motions of a particle in a fluid is of interest for the theory of Brownian motion of particles with rotational degrees of freedom. In order to investigate the general features of this coupling, one may use the hydrodynamical description. Starting from the linearized, steady state Navier-Stokes equation, one finds the force and torque acting on a rigid particle in a fluid as linear functions of both the linear and angular velocities of the particle [1]. The consequences of such a linear type of coupling for the Brownian motion have already been studied by Condiff and Dahler [2]. This linear coupling, however, does not exist for particles with three mutually perpendicular symmetry planes [1], in particular not for rigid ellipsoids and spheres. It is the purpose of this paper to show that a different, bi-linear type of coupling between translational and rotational motions exists (in accordance with hydrodynamics) even in the case of spherical particles. Therefore we may confine our attention to this case. This second type of coupling emerges from the quadratic inertia term of the exact steady state Navier-Stokes equation. It consists in a "transverse" force proportional to the cross product of the linear velocity and the angular velocity of the rigid sphere. The transverse force causes a particle to roll aside if its angular velocity is not parallel to its velocity. Experimentally the existence of such a force has been observed long ago (with cannon balls, tennis and golf balls) and it is known as Magnus effect [3–6]. By the way, for an infinitely long cylinder (2 dimensional problem, multiply connected space) a transverse force may even exist if the cylinder rotates in a streaming inviscid fluid [6]. This, however, is not true for the three dimensional problem, e. g. for the force acting on a sphere. Here one really has to consider a viscous fluid (as described by the Navier-Stokes equation) in order to find a force on a particle [7].

A transverse force also acts on a rough sphere (with radius $R \ll l$: mean free path) in a dilute gas of rough spheres [7a].

The influence of the transverse force on the Brownian motion of rotating spheres has already been studied by the author [8].

§ 1. General Remarks

Let us consider a rigid sphere simultaneously translating and rotating ("spinning") in an incompressible viscous fluid which is at rest at infinity. The linear velocity of the center of the sphere is denoted by $-\boldsymbol{V}$. The angular velocity of the sphere with respect to its center is called $\boldsymbol{\Omega}$. Both $\boldsymbol{V}$ and $\boldsymbol{\Omega}$ shall be constant. We introduce a coordi-

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nate system where the center of the sphere is at rest (and coincides with the origin). Then the streaming velocity $\boldsymbol{v}$ of the fluid is $\boldsymbol{V}$ at infinity. The radius R of the sphere shall be large compared with the mean free path in the fluid. Hence in order to calculate the force and torque acting on the sphere (in steade state) one has to start from the Navier-Stokes equation (with the time derivative put equal to zero) and a no-slip boundary condition may be used.

We say a coupling of translational and rotational motions of a particle (in a fluid) occurs if the force or/and the torque depend on both linear and angular velocities. Neglecting the quadratic inertia term in the Navier-Stokes equation one in general finds a linear type of coupling with the force and torque linear in $\boldsymbol{V}$ and $\boldsymbol{\Omega}$. This coupling, however, vanishes for particles with three mutually perpendicular symmetry planes (Ref. [1], p. 187). Hence for a sphere, Stokes' friction force and torque (which are derived from the linearized Navier-Stokes equation) cannot show a coupling of translational and rotational motions. Yet one may expect such a coupling to occur if one considers force and torque expressions non-linear with respect to $\boldsymbol{V}$ and $\boldsymbol{\Omega}$. In particular, a transverse force

$$\boldsymbol{K}^{\text{trans}} \sim \boldsymbol{V} \times \boldsymbol{\Omega} \tag{1.1}$$

might act on sphere immersed in a fluid [9]. It is the purpose of this paper to show that, indeed, such a force can be derived from the Navier-Stokes equation if the quadratic inertia term is taken into account.

Two remarks concerning the transverse force can be made at once:

i) In hydrodynamical equations, transport coefficients like the viscosity η occur in equations, where two quantities with different time reversal behavior are linked. As $\boldsymbol{K}$ and $\boldsymbol{V} \times \boldsymbol{\Omega}$ are both even under time reversal, η will not appear in (1.1). (In general only even powers of η might occur.) But this does not imply the possibility to restrict oneself to inviscid fluids. In this case the no-slip boundary condition cannot be fulfilled.

ii) The constant in (1.1) must have the dimension of a mass. The only mass linked with our problem is the mass m_{f} of the fluid contained in a volume equal to that of the sphere:

$$m_{\text{f}} = \tfrac{4}{3} \pi R^3 \varrho \tag{1.2}$$

[9] A torque $\boldsymbol{M} \sim \boldsymbol{V} \times \boldsymbol{\Omega}$ cannot exist since $\boldsymbol{M}$ and $\boldsymbol{V} \times \boldsymbol{\Omega}$ have different parities.

where ϱ is the mass density of the fluid. Hence the transverse force can be written as

$$\boldsymbol{K}^{\text{trans}} = \gamma\, m_{\text{f}}\, \boldsymbol{V} \times \boldsymbol{\Omega} \tag{1.3}$$

where γ is a dimensionless quantity.

In general γ might depend on V^2 and Ω^2. Here, however, we confine ourselves to the coupling of translational and rotational motions of lowest order in $\boldsymbol{V}$ and $\boldsymbol{\Omega}$, where γ is a constant.

§ 2. Navier-Stokes Equation

Introducing the dimensionless vector $\boldsymbol{x} = R^{-1} \tilde{\boldsymbol{x}}$ where R is the radius of the sphere and $\tilde{\boldsymbol{x}}$ is the proper position vector (with $\tilde{\boldsymbol{x}} = 0$ corresponding to the center of the sphere), one has as the steady state Navier-Stokes equation

$$\varrho\, \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \nabla p = \eta\, R^{-1} \Delta \boldsymbol{v} \,. \tag{2.1}$$

Here $\nabla = \partial / \partial \boldsymbol{x}$ is the dimensionless Nabla-operator and $\Delta = \nabla \cdot \nabla$ is the dimensionless Laplacian. The fluid is assumed to be incompressible, hence the divergence of the velocity vanishes

$$\nabla \cdot \boldsymbol{v} = \operatorname{div} \boldsymbol{v} = 0 \,. \tag{2.2}$$

The density ϱ occuring in the inertia term of (2.1) is constant. An equation for p can be obtained by taking the divergence of (2.1) and using (2.2)

$$\varrho \operatorname{div}(\boldsymbol{v} \cdot \nabla \boldsymbol{v}) + \Delta p = 0 \,. \tag{2.3}$$

The pressure p and the flow velocity $\boldsymbol{v}$ will depend on the dimensionles space coordinates $\boldsymbol{x}$ and on $\boldsymbol{V}$ and $\boldsymbol{\Omega}$ as parameters

$$p = p(\boldsymbol{x}; \boldsymbol{V}, \boldsymbol{\Omega}) \,; \quad \boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x}; \boldsymbol{V}, \boldsymbol{\Omega}) \,. \tag{2.4}$$

We now expand p and $\boldsymbol{v}$ in powers of $\boldsymbol{V}$ and $\boldsymbol{\Omega}$

$$p = \sum_{i,j=0} p^{(ij)} , \qquad \boldsymbol{v} = \sum_{i,j=0} \boldsymbol{v}^{(ij)} , \tag{2.5}$$

where $p^{(ij)}$ and $\boldsymbol{v}^{(ij)}$ are of i-th power in Cartesian components of $\boldsymbol{V}$ and of j-th power in those of $\boldsymbol{\Omega}$.

For $\boldsymbol{V} = 0$ and $\boldsymbol{\Omega} = 0$ the fluid shall be at rest and the pressure is constant, thus we have

$$\boldsymbol{v}^{(00)} = 0 \quad \text{and} \quad p^{(00)} = p_\infty = \text{const} \,. \tag{2.6}$$

The (ij)-part of the Navier-Stokes equation (2.1) reads

$$\varrho \sum_{\substack{i'j' \\ i''j''}} \boldsymbol{v}^{(i'j')} \cdot \nabla \boldsymbol{v}^{(i''j'')} + \nabla p^{(ij)} = \eta\, R^{-1} \Delta \boldsymbol{v}^{(ij)} \tag{2.7}$$

where

$$i' + i'' = i ; \quad j' + j'' = j ; \quad 0 \leqq i', i'' < i ; \quad 0 \leqq j', j'' < j .$$

The corresponding incompressibility condition is

$$\operatorname{div} \boldsymbol{v}^{(ij)} = 0\,. \tag{2.8}$$

In this way the quadratic Navier-Stokes equation has been reduced to an infinite set of linear differential equations for $\boldsymbol{v}^{(ij)}$. These may be solved by an iteration procedure since the quadratic term of the (ij)-th equation is a known inhomogeneous term if all $\boldsymbol{v}^{(i'j')}$ with $i'<i$ and $j'<j$ already have been determined. It is not the purpose of this paper to tackle the question of convergence of such an expansion since we are interested in finding the coupling of translational and rotational motions in lowest order in $\boldsymbol{V}$ and $\boldsymbol{\Omega}$.

The first two equations of (2.7) and (2.8) correspond to the linearized Navier-Stokes equation supplemented by the pertaining incompressibility condition

$$\begin{aligned} \nabla p^{(10)} &= \eta\, R^{-1} \Delta \boldsymbol{v}^{(10)}, \quad \operatorname{div} \boldsymbol{v}^{(10)} = 0;\\ \nabla p^{(01)} &= \eta\, R^{-1} \Delta \boldsymbol{v}^{(01)}, \quad \operatorname{div} \boldsymbol{v}^{(01)} = 0. \end{aligned} \tag{2.9}$$

Here the inertia term drops out. Translational and rotational motions are not coupled. The desired coupling occurs in lowest order in the "bi-linear equation" (i. e. bi-linear in $\boldsymbol{V}, \boldsymbol{\Omega}$)

$$\varrho(\boldsymbol{v}^{(10)} \cdot \nabla \boldsymbol{v}^{(01)} + \boldsymbol{v}^{(01)} \cdot \nabla \boldsymbol{v}^{(10)}) + \nabla p^{(11)} = \eta\, R^{-1} \Delta \boldsymbol{v}^{(11)}. \tag{2.10}$$

In accordance with (2.8) $\boldsymbol{v}^{(11)}$ has to obey the incompressibility condition

$$\operatorname{div} \boldsymbol{v}^{(11)} = 0\,. \tag{2.11}$$

After having solved (2.10) subject to (2.11) and to appropriate boundary conditions, we shall see that $p^{(11)}$ and

$$p_{\mu\nu}^{(11)} = -\eta\, R^{-1} \left(\frac{\partial v_\mu^{(11)}}{\partial x_\nu} + \frac{\partial v_\nu^{(11)}}{\partial x_\mu} \right) \tag{2.12}$$

will give rise to a transverse force of the type (1.3). Therefore, we confine our attention to the solution of (2.10). In (2.12) and in the following, Cartesian components of vectors and 2nd rank tensors are denoted by Greek subscripts. The summation convention will be used for these.

It is convenient to express the space dependence of p and $\boldsymbol{v}$ in terms of Cartesian multipole potentials. A list of the scalars and vectors which may be constructed from the multipoles and the Cartesian components of $\boldsymbol{V}$ and $\boldsymbol{\Omega}$ shows, that in order to determine the bi-linear pressure $p^{(11)}$ and velocity $\boldsymbol{v}^{(11)}$, one has actually to find one and three scalar coefficients respectively as functions of the distance. This point shall be discussed in the next section.

§ 3. Expansion of Pressure and Velocity

The space dependence of the pressure and the velocity shall be described in terms of the multiple potentials defined by

$$X_{\mu_1 \ldots \mu_l} = (-1)^l \frac{\partial^l r^{-1}}{\partial x_{\mu_1} \ldots \partial x_{\mu_l}}. \tag{3.1}$$

Here r is the distance of a point from the center of the sphere measured in multiples of the radius R ($r^2 = \boldsymbol{x} \cdot \boldsymbol{x}$; $r^2 = 1$ surface of the sphere). Since $\Delta r^{-1} = 0$, the multipoles are solutions of the Laplace equation. The l-th multipole is a function homogeneous in r^{-l-1} and is proportional to the symmetric irreducible tensor of l-th rank that can be constructed from the unit vector $\hat{\boldsymbol{x}} = r^{-1} \boldsymbol{x}$. For further properties of the multipole potentials see the appendix. The first multipoles are

$$\begin{aligned} X &= r^{-1},\\ X_\mu &= r^{-3} x_\mu\,,\\ X_{\mu\nu} &= 3\, r^{-5} (x_\mu x_\nu - \tfrac{1}{3} r^2 \delta_{\mu\nu})\,. \end{aligned} \tag{3.2}$$

The scalars and vectors which may be constructed from the multipole potentials and the Cartesian Components of $\boldsymbol{V}$ and $\boldsymbol{\Omega}$ are listed in Table 1 up to quadratic terms in $\boldsymbol{V}$ and $\boldsymbol{\Omega}$. The column P states the parity (P: $\boldsymbol{x}$, $\boldsymbol{V}, \boldsymbol{\Omega} \rightarrow -\boldsymbol{x}$, $-\boldsymbol{V}$, $\boldsymbol{\Omega}$) of the expansion tensors. For convenience the scalars, pseudo scalars, polar and axial vectors are denoted by Φ, Ψ; $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ respectively and distinguished by consecutive superscripts. The symmetric irreducible (traceless) part of a dyadic constructed from the Cartesian components of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is denoted by

$$\overline{(\boldsymbol{a}\,\boldsymbol{b})}_{\mu\nu} = \overline{a_\mu b_\nu} = \tfrac{1}{2} (a_\mu b_\nu + b_\mu a_\nu) - \tfrac{1}{3} \boldsymbol{a} \cdot \boldsymbol{b}\, \delta_{\mu\nu}; \tag{3.3}$$

$\varepsilon_{\mu\nu\lambda}$ is the anti-symmetric isotropic 3rd rank tensor with the property $\varepsilon_{\mu\nu\lambda} a_\nu b_\lambda = (\boldsymbol{a} \times \boldsymbol{b})_\mu$. The expansion scalars and vectors chosen are linearly independent. This may easily be seen from Table 1. Tensors with different parity and different power in $\boldsymbol{V}$ and $\boldsymbol{\Omega}$ are independent per se. Those with equal parity and equal power in $\boldsymbol{V}$ and $\boldsymbol{\Omega}$ either contain different multipole potentials (e. g. $\boldsymbol{\Phi}^{(1)}$ and $\boldsymbol{\Phi}^{(3)}$; $\boldsymbol{\Phi}^{(7)}$ and $\boldsymbol{\Phi}^{(8)}$) or different irreducible parts of tensors constructed from $\boldsymbol{V}$ and $\boldsymbol{\Omega}$, e. g. $\boldsymbol{\Phi}^{(8)}$ contains the anti-symmetric part of the 2nd rank tensor $\boldsymbol{V\,\Omega}$, $\boldsymbol{\Phi}^{(9)}$ the symmetric ireducible part. Furthermore the expansion tensors are complete in the sense that any scalar or vector which can be constructed from the Cartesian components of $\boldsymbol{x}$, $\boldsymbol{V}$ and $\boldsymbol{\Omega}$ (up to second power in $\boldsymbol{V}$, $\boldsymbol{\Omega}$) can be written as a linear combination of the expansion scalars and vectors listed in Table 1. Expansion tensors quadratic in $\boldsymbol{V}$ or $\boldsymbol{\Omega}$ alone will not be needed in the following but are listed for completeness.

Power in V	$\boldsymbol{\Omega}$	Scalars	P	Vectors	P
0	0	$\Phi^{(0)}=1$	+	$\Phi_\mu^{(0)}=X_\mu$	−
1	0			$\Phi_\mu^{(1)}=V_\mu$	−
		$\Phi^{(1)}=\boldsymbol{V}\cdot\boldsymbol{X}$	+	$\Psi_\mu^{(1)}=(\boldsymbol{V}\times\boldsymbol{X})_\mu$	+
				$\Phi_\mu^{(2)}=X_{\mu\nu}\,V_\nu$	−
0	1			$\Psi_\mu^{(2)}=\Omega_\mu$	+
		$\Psi^{(1)}=\boldsymbol{\Omega}\cdot\boldsymbol{X}$	−	$\Phi_\mu^{(3)}=(\boldsymbol{\Omega}\times\boldsymbol{X})_\mu$	−
				$\Psi_\mu^{(3)}=X_{\mu\nu}\,\Omega_\nu$	+
2	0	$\Phi^{(2)}=V^2$	+	$\Phi_\mu^{(4)}=V^2\,X_\mu$	−
		$\Phi^{(3)}=V_\mu\,X_{\mu\nu}\,V_\nu$	+	$\Phi_\mu^{(5)}=\overline{V_\mu\,V_\nu}\,X_\nu$	−
				$\Phi_\mu^{(6)}=X_{\mu\nu\lambda}\,V_\nu\,V_\lambda$	−
1	1	$\Psi^{(2)}=\boldsymbol{V}\cdot\boldsymbol{\Omega}$	−	$\Phi_\mu^{(7)}=(\boldsymbol{V}\times\boldsymbol{\Omega})_\mu$	−
		$\Phi^{(1)}=\boldsymbol{X}\cdot(\boldsymbol{V}\times\boldsymbol{\Omega})$	+	$\Psi_\mu^{(4)}=\boldsymbol{V}\cdot\boldsymbol{\Omega}\,X_\mu$	+
		$\Psi^{(3)}=V_\mu\,X_{\mu\nu}\,\Omega_\nu$	−	$\Psi_\mu^{(5)}=\overline{V_\mu\,\Omega_\nu}\,X_\nu$	+
				$\Phi_\mu^{(8)}=X_{\mu\nu}\,(\boldsymbol{V}\times\boldsymbol{\Omega})_\nu$	−
				$\Phi_\mu^{(9)}=\varepsilon_{\mu\varrho\tau}\,X_{\varrho\lambda}\,\overline{\Omega_\lambda\,V_\tau}$	−
				$\Psi_\mu^{(6)}=X_{\mu\nu\lambda}\,V_\nu\,\Omega_\lambda$	+
0	2	$\Phi^{(1)}=\Omega^2$	+	$\Phi_\mu^{(10)}=\Omega^2\,X_\mu$	−
		$\Phi^{(6)}=\Omega_\mu\,X_{\mu\nu}\,\Omega_\nu$	+	$\Phi_\mu^{(11)}=\overline{\Omega_\mu\,\Omega_\nu}\,X_\nu$	−
				$\Phi_\mu^{(12)}=X_{\mu\nu\lambda}\,\Omega_\nu\,\Omega_\lambda$	−

Table 1. The first expansion scalars and vectors constructed from the multipole potentials and the components of $\boldsymbol{V}$ and $\boldsymbol{\Omega}$.

Now we are ready to make an ansatz for the pressure p and the velocity $\boldsymbol{v}$ in terms of the expansion tensors. The pressure has positive parity, thus scalars with positive parity only can occur in its expansion. From Table 1 we find the following ansatz for that part of the expansion for p we are interested in

$$p^{(10)}=A_1\,\Phi^{(1)};\quad p^{(01)}=0;\quad p^{(11)}=A_4\,\Phi^{(4)}. \tag{3.4}$$

The coefficients A_i are functions of the distance r. The velocity has negative parity. Therefore one has

$$\boldsymbol{v}^{(10)}=\alpha_1\,\boldsymbol{\Phi}^{(1)}+\alpha_2\,\boldsymbol{\Phi}^{(2)};\quad \boldsymbol{v}^{(01)}=\alpha_3\,\boldsymbol{\Phi}^{(3)}; \tag{3.5}$$

$$\boldsymbol{v}^{(11)}=\alpha_7\,\boldsymbol{\Phi}^{(7)}+\alpha_8\,\boldsymbol{\Phi}^{(8)}+\alpha_9\,\boldsymbol{\Phi}^{(9)}. \tag{3.6}$$

Again the scalar coefficients α_i are functions of r.

Next we have to state the boundary conditions for the coefficients A_i $(i=1,4)$ and α_k $(k=1, 2, 3; 7, 8, 9)$.

a) The pressure shall be constant at infinity. This implies $p^{(ij)}$ $(i\,j \neq 0\,0)$ to vanish for $r\to\infty$. In particular one has to require (since $\Phi^{(1)}$ and $\Phi^{(4)}$ are proportional to r^{-2})

$$r^{-2}\,A_1(r)\to 0 \text{ and } r^{-2}\,A_4(r)\to 0 \text{ for } r\to\infty. \tag{3.7}$$

b) At infinity, the velocity $\boldsymbol{v}$ of fluid equals $\boldsymbol{V}$. Hence $\boldsymbol{v}^{(10)}$ tends to $\boldsymbol{V}$ and all other $\boldsymbol{v}^{(ij)}$ have to vanish for $r\to\infty$. In particular one has

$$\alpha_1(\infty)=1, \tag{3.8}$$

$$r^{-3}\,\alpha_2(r)\to 0,\quad r^{-2}\,\alpha_3(r)\to 0 \quad \text{for } r\to\infty, \tag{3.9}$$

$$\alpha_7(\infty)=0, \tag{3.10}$$

$$r^{-3}\,\alpha_8(r)\to 0,\quad r^{-3}\,\alpha_9(r)\to 0 \quad \text{for } r\to\infty. \tag{3.11}$$

c) There shall be no velocity slip at the surface of the sphere $(r=1)$. Hence one has $\boldsymbol{v}=R\,\boldsymbol{\Omega}\times\boldsymbol{x}$ for $r=1$. This is fulfilled if

$$\alpha_3(1)=R \quad \text{and} \tag{3.12}$$

$$\alpha_k(1)=0 \quad \text{for} \quad k\neq 3. \tag{3.13}$$

Solution of the Eqs. (2.9), corresponding to the linearized Navier-Stokes equation with the pertaining boundary conditions, yields the well known results

$$A_1=\tfrac{3}{2}\,\eta\,R^{-1}, \tag{3.14}$$

$$\alpha_1=(1-r^{-1});\quad \alpha_2=\tfrac{1}{4}\,(1-r^2);\quad \alpha_3=R, \tag{3.15}$$

which in turn lead to the Stokes friction force and torque [1, 6, 7].

In order to determine A_4 and α_7, α_8, α_9 as functions of r, one has to solve the "bi-linear" Navier-Stokes equation (2.10). Since A_4 and α_7 only are needed for the calculation of the transverse force we focus our attention on the evaluation of these coefficients and then mention the results for α_8 and α_9 briefly.

§ 4. Calculation of the "bi-linear" Pressure and Velocity

The unknown pressure $p^{(11)}$ and velocity $\boldsymbol{v}^{(11)}$ are uniquely determined by the "bi-linear" Navier-Stokes equation (2.10), the incompressibility condition (2.11) and the boundary conditions stated above.

Before dealing with Eq. (2.10) we observe that due to

$$\operatorname{div} \boldsymbol{v}^{(11)} = (r^2 \alpha_7' + 2 r^{-1} \alpha_8')\, \Phi^{(4)}, \qquad (4.1)$$

the incompressibility condition (2.11) implies

$$r^2 \alpha_7' + 2 r^{-1} \alpha_8' = 0\,. \qquad (4.2)$$

The prime denotes differentiation with respect to r.

To find the four coefficients A_4, α_7, α_8, α_9 specifying $p^{(11)}$ and $\boldsymbol{v}^{(11)}$ from Eqs. (2.10), (4.2) and the boundary conditions mentioned earlier, we now proceed in three steps:

1) The inhomogeneity term

$$\mathbf{Q}^{(11)} = \varrho (\boldsymbol{v}^{(10)} \cdot \nabla \boldsymbol{v}^{(01)} + \boldsymbol{v}^{(01)} \cdot \nabla \boldsymbol{v}^{(10)}) \qquad (4.3)$$

of Eq. (2.10) is evaluated from the known functions $\boldsymbol{v}^{(10)}$ and $\boldsymbol{v}^{(01)}$ [cf. (3.5) and (3.15)]. Use of some properties of the multipole potentials (see appendix) by a straightforward calculation yields

$$\boldsymbol{v}^{(10)} \cdot \nabla \boldsymbol{v}^{(01)} = R\{2 r^{-6} \alpha_2 \boldsymbol{\Phi}^{(7)} + (\alpha_1 + r^{-3} \alpha_2)\, \boldsymbol{\Phi}\}\,. \qquad (4.4)$$

Here $\boldsymbol{\Phi}$ is an abbreviation for the vector

$$\Phi_\mu = \varepsilon_{\mu\nu\lambda} X_{\nu\varrho} V_\varrho \Omega_\lambda\,, \qquad (4.5)$$

which may be expressed by $\boldsymbol{\Phi}^{(8)}$ and $\boldsymbol{\Phi}^{(9)}$

$$\boldsymbol{\Phi} = -\tfrac{1}{2} \boldsymbol{\Phi}^{(8)} + \boldsymbol{\Phi}^{(9)}. \qquad (4.6)$$

Likewise one finds

$$\begin{aligned} \boldsymbol{v}^{(01)} \cdot \nabla \boldsymbol{v}^{(10)} &= R\, r^{-3}\, \boldsymbol{\Omega} \cdot \mathfrak{L}\, \boldsymbol{v}^{(10)} \\ &= R\, r^{-3} \alpha_2 (\boldsymbol{\Phi}^{(8)} - \boldsymbol{\Phi})\,. \end{aligned} \qquad (4.7)$$

The differential operator $\mathfrak{L} = \boldsymbol{x} \times (\partial / \partial \boldsymbol{x})$ only acts on the angle dependent part of a function; its operation on a multiple potential is stated in the appendix. Adding (4.4) and (4.7) and using (4.6) one finds

$$\mathbf{Q}^{(11)} = \varrho\, R (a_7 \boldsymbol{\Phi}^{(7)} + a_8 \boldsymbol{\Phi}^{(8)} + a_9 \boldsymbol{\Phi}^{(9)})\,, \qquad (4.8)$$

where

$$\begin{aligned} a_7 &= 2 r^{-6} \alpha_2 &&= \tfrac{1}{2} r^{-6} - \tfrac{1}{2} r^{-4}, \\ a_8 &= r^{-3} \alpha_2 - \tfrac{1}{2} \alpha_1 &&= \tfrac{1}{4} r^{-3} + \tfrac{1}{4} r^{-1} - \tfrac{1}{2}, \\ a_9 &= \alpha_1 &&= 1 - r^{-1}. \end{aligned} \qquad (4.9)$$

2) The pressure $p^{(11)}$ is determined (apart from a constant) by the differential equation corresponding to (2.3)

$$\operatorname{div} \mathbf{Q}^{(11)} + \Delta p^{(11)} = 0\,. \qquad (4.10)$$

Similarly to (4.1) one has

$$\operatorname{div} \mathbf{Q}^{(11)} = \varrho\, R (r^2 a_7' + 2 r^{-1} a_8')\, \Phi^{(4)}. \qquad (4.11)$$

On account of

$$\Delta p^{(11)} = r^2 (r^{-2} A_4')'\, \Phi^{(4)}\,, \qquad (4.12)$$

Eq. (4.10) leads to a second order differential equation for A_4 containing an inhomogeneous term which is known from (4.11) and (4.9). Use of the boundary condition (3.7) leads to

$$A_4 = \varrho\, R (C_4 - \tfrac{3}{8} r^{-1} + \tfrac{1}{4} r^{-3})\,. \qquad (4.13)$$

The constant C_4 will be determined later.

3) The inhomogeneity term $\mathbf{Q}^{(11)}$ of the Navier-Stokes equation (2.10) and the pressure $p^{(11)}$ (apart from the constant C_4) being known, we now are ready to evaluate $\boldsymbol{v}^{(11)}$. By use of

$$\nabla p^{(11)} = \tfrac{1}{3} r^{-3} A_4'\, \boldsymbol{\Phi}^{(7)} + (\tfrac{1}{3} r A_4' - A_4)\, \boldsymbol{\Phi}^{(8)} \qquad (4.14)$$

[with A_4 from (4.13)] and with the representation following from (3.6)

$$\begin{aligned} \Delta \boldsymbol{v}^{(11)} = {} & (\alpha_7'' + 2 r^{-1} \alpha_7')\, \boldsymbol{\Phi}^{(7)} + (\alpha_8'' - 4 r^{-1} \alpha_8')\, \boldsymbol{\Phi}^{(8)} \\ & + (\alpha_9'' - 4 r^{-1} \alpha_9')\, \boldsymbol{\Phi}^{(9)}, \end{aligned} \qquad (4.15)$$

Eq. (2.10) may be written down explicitly. Equating coefficients of the three linearly independent vectors $\boldsymbol{\Phi}^{(7)}$, $\boldsymbol{\Phi}^{(8)}$, $\boldsymbol{\Phi}^{(9)}$ on both sides one obtains three uncoupled 2nd order differential equations for α_7, α_8, α_9. These equations can be solved with the boundary conditions (3.10, 11, 13). The coefficient α_8 then still contains the constant C_4 occuring in (4.13); it is determined by the incompressibility condition (4.2).

Since we are mainly interested in the transverse force which is solely determined by $\alpha_7'(1)$ and $A_4(1)$ we apply a different procedure which is more convenient for the evaluation of $\alpha_7(r)$ and C_4. To this end we dot (2.10) by $\boldsymbol{x}$. The resulting equation is

$$\boldsymbol{x} \cdot \mathbf{Q}^{(11)} + \boldsymbol{x} \cdot \nabla p^{(11)} = \eta\, R^{-1} \boldsymbol{x} \cdot \Delta \boldsymbol{v}^{(11)}, \qquad (4.16)$$

all terms of which are proportional to the scalar $\Phi^{(4)}$.

This is seen from the following relations:

$$\boldsymbol{x} \cdot \mathbf{Q}^{(11)} = \varrho\, R (r^3 a_7 + 2 a_8)\, \Phi^{(4)}, \qquad (4.17)$$

$$\boldsymbol{x} \cdot \nabla p^{(11)} = r \frac{\partial p^{(11)}}{\partial r} = (r A_4' - 2 A_4)\, \Phi^{(4)}, \qquad (4.18)$$

$$\boldsymbol{x} \cdot \Delta \boldsymbol{v}^{(11)} = \Delta \boldsymbol{x} \cdot \boldsymbol{v}^{(11)} = r^2 [r^{-2} (r^3 \alpha_7 + 2 \alpha_8)']'\, \Phi^{(4)}. \qquad (4.19)$$

Eliminating α_8' from (4.19) by use of (4.2) one finds

$$\boldsymbol{x} \cdot \Delta \boldsymbol{v}^{(11)} = 3 r^2 \alpha_7'\, \Phi^{(4)}. \qquad (4.20)$$

Thus the following 1st order differential for α_7 is obtained from (4.16):

$$\varrho R(r^3 \alpha_7 + 2 \alpha_8) + r A_4' - A_4 = 3 \eta R^{-1} r^2 \alpha_7'. \quad (4.21)$$

According to (4.9) and (4.13) this equations reads

$$\alpha_7' = \frac{\varrho R^2}{3 \eta} [-(1+2 C_4) r^{-2} + \tfrac{9}{8} r^{-3} - \tfrac{1}{4} r^{-5}]. \quad (4.22)$$

The solution of (4.22) is

$$\alpha_7 = \frac{\varrho R^2}{3 \eta} [(1+2 C_4) r^{-1} - \tfrac{9}{16} r^{-2} + \tfrac{1}{16} r^{-4}] + c_7. \quad (4.23)$$

The constant c_7 has to vanish due to the boundary condition $\alpha_7(\infty) = 0$. In order to satisfy the boundary condition at the surface of the sphere $\alpha_7(1) = 0$ we have to put

$$C_4 = -\tfrac{1}{4}. \quad (4.24)$$

Hence the final result for α_7 is

$$\alpha_7 = (\varrho R^2/\eta)(\tfrac{1}{6} r^{-1} - \tfrac{3}{16} r^{-2} + \tfrac{1}{48} r^{-4}). \quad (4.25)$$

Although α_8 and α_9 are not needed for the calculation of the transverse force the results (obtained by using the above mentioned solution procedure) are given for completeness:

$$\alpha_8 = (\varrho R^2/\eta)(\tfrac{1}{24} r^2 - \tfrac{3}{16} r + \tfrac{3}{16} - \tfrac{1}{24} r^{-1}), \quad (4.26)$$

$$\alpha_9 = (\varrho R^2/\eta)(-\tfrac{1}{6} r^2 + \tfrac{1}{4} r - \tfrac{1}{12}). \quad (4.27)$$

By the way, the viscosity η appearing in the denominator clearly indicates that the solutions found do not exist for an ideal fluid ($\eta = 0$). Nevertheless η will not show up in the transverse force which will be evaluated in the following section.

§ 5. Transverse Force

Both the hydrostatic pressure $p^{(11)}$ and the friction pressure $p_{\mu\nu}^{(11)}$ [see (2.12)] give a contribution to the transverse force $\mathbf{K}^{\text{tr}}$. The first one is

$$\mathbf{K}^{\text{tr, p}} = -\int \hat{\boldsymbol{x}}\, p^{(11)}\, \mathrm{d}^2 o, \quad (5.1)$$

where $\hat{\boldsymbol{x}} = r^{-1} \boldsymbol{x}$ is a radial unit vector and $\mathrm{d}^2 o$ is the surface element of the sphere. Substituting the ansatz (3.4) into (5.1) and carrying out the integration one finds

$$\mathbf{K}^{\text{tr, p}} = -\tfrac{4}{3} \pi R^2 A_4(1)\, \boldsymbol{V} \times \boldsymbol{\Omega}. \quad (5.2)$$

After (4.13) and (4.24) one has $A_4(1) = -\frac{3}{8} \varrho R$ and thus

$$\mathbf{K}^{\text{tr, p}} = \tfrac{3}{8} m_{\text{f}}\, \boldsymbol{V} \times \boldsymbol{\Omega}. \quad (5.3)$$

For m_{f} see (1.2).

The contribution from the friction pressure is

$$K_\mu^{\text{tr, f}} = -\int p_{\mu\nu}^{(11)}\, \hat{x}_\nu\, \mathrm{d}^2 o = \eta R^{-1} \int \left(\hat{x}_\nu \frac{\partial v_\mu^{(11)}}{\partial x_\nu} + \hat{x}_\nu \frac{\partial v_\nu^{(11)}}{\partial x_\mu} \right) \mathrm{d}^2 o. \quad (5.4)$$

By use of the ansatz (3.6) and of (2.11), Eq. (5.4) takes the form

$$\mathbf{K}^{\text{tr, f}} = 4 \pi R \eta\, \alpha_7'(1)\, \boldsymbol{V} \times \boldsymbol{\Omega}. \quad (5.5)$$

From (4.22) and (4.24) one infers $\eta\, \alpha_7'(1) = \frac{1}{8} \varrho R^2$.

Hence the total transverse force is

$$\mathbf{K}^{\text{tr}} = \mathbf{K}^{\text{tr, p}} + \mathbf{K}^{\text{tr, f}} = \tfrac{3}{4} m_{\text{f}}\, \boldsymbol{V} \times \boldsymbol{\Omega}. \quad (5.6)$$

This is a force of type (1.3) with $\gamma = \frac{3}{4}$. It remains unchanged if the center of the sphere moves with velocity $-\boldsymbol{V}$ and the fluid is at rest far away from the sphere.

Appendix: Cartesian Multipole Potentials

The l-th Cartesian multipole potential (l-pole) is a l-th rank ireducible tensor defined by

$$X_{\mu_1 \ldots \mu_l} = (-1)^l \frac{\partial^l r^{-1}}{\partial x_{\mu_1} \ldots \partial x_{\mu_l}} = -\frac{\partial X_{\mu_1 \ldots \mu_{l-1}}}{\partial x_{\mu_l}} \quad (l = 0, 1, 2, \ldots). \quad (\text{A.1})$$

These tensors are solutions of the Laplace equation

$$\Delta X_{\mu_1 \ldots \mu_l} = 0 \quad (\text{A.2})$$

which vanish for $r \to \infty$. They are symmetric in all subscripts and have zero traces, i. e.

$$X_{\mu_1 \ldots \mu_{l-2} \mu\mu} = 0. \quad (\text{A.3})$$

Thus, a l-pole has not 3^l but only $(2l+1)$ independent components. The l-pole is homogeneous in r^{-1-l} and proportional to the irreducible tensor $\overline{\hat{x}_{\mu_1} \ldots \hat{x}_{\mu_l}}$ of rank l which can be constructed from the unit vector $\hat{\boldsymbol{x}} = r^{-1} \boldsymbol{x}$:

$$X_{\mu_1 \ldots \mu_l} = r^{-1-l} (2l-1)!!\, \overline{\hat{x}_{\mu_1} \ldots \hat{x}_{\mu_l}}, \quad (\text{A.4})$$

where $(2l-1)!! = 1 \cdot 3 \cdot 5 \ldots (2l-1)$.

The normalization constant of the irreducible l-th rank tensor (denoted by $\overline{}$) is chosen in such a way that the coefficient of the term $x_{\mu_1} \ldots x_{\mu_l}$ contained in $\overline{x_{\mu_1} \ldots x_{\mu_l}}$ equals 1. This implies

$$\hat{x}_{\mu_1} \ldots \hat{x}_{\mu_l}\, \overline{\hat{x}_{\mu_1} \ldots \hat{x}_{\mu_l}} = \frac{l!}{(2l-1)!!}. \quad (\text{A.5})$$

Multiplication of a l-pole with $\boldsymbol{x}$ leads to a $(l+1)$-pole and a sum of $(l-1)$-poles:

$$x_{\mu_{l+1}} X_{\mu_1 \dots \mu_l} = (2l+1)^{-1} [r^2 X_{\mu_1 \dots \mu_{l+1}} + l(2l-1) \Delta^{(l)}_{\mu_1 \dots \mu_l, \mu_{l+1} \nu_1 \dots \nu_{l-1}} X_{\nu_1 \dots \nu_{l-1}}]. \quad \text{(A.6)}$$

Here $\Delta^{(l)}_{\dots,\dots}$ is an isotropic tensor of rank $2l$, which, applied on an arbitrary l-th rank tensor, projects on the irreducible part of this tensor (denoted by $\overline{}$):

$$\Delta^{(l)}_{\mu_1 \dots \mu_l, \nu_1 \dots \nu_l} a_{\nu_1 \dots \nu_l} = \overline{a_{\mu_1 \dots \mu_l}}. \quad \text{(A.7)}$$

The $\Delta^{(l)}$-tensors have the general properties:

$$\Delta^{(l)}_{\mu_1 \dots \mu_l, \lambda_1 \dots \lambda_l} \Delta^{(l)}_{\lambda_1 \dots \lambda_l, \nu_1 \dots \nu_l} = \Delta^{(l)}_{\mu_1 \dots \mu_l, \nu_1 \dots \nu_l} \quad \text{and} \quad \Delta^{(l)}_{\mu_1 \dots \mu_l, \mu_1 \dots \mu_l} = 2l+1. \quad \text{(A.8, 9)}$$

The first 3 of these $\Delta^{(l)}$-tensors are

$$\Delta^{(0)} = 1; \quad \Delta^{(1)}_{\mu\nu} = \delta_{\mu\nu}; \quad \Delta^{(2)}_{\mu\nu,\mu'\nu'} = \tfrac{1}{2} (\delta_{\mu\mu'} \delta_{\nu\nu'} + \delta_{\nu\mu'} \delta_{\mu\nu'}) - \tfrac{1}{3} \delta_{\mu\nu} \delta_{\mu'\nu'}. \quad \text{(A.10)}$$

Putting $\mu_{l+1} = \mu_l$ (with automatic summation) in (A.6) one obtains the relation

$$x_{\mu l} X_{\mu_1 \dots \mu_l} = l X_{\mu_1 \dots \mu_{l-1}}. \quad \text{(A.11)}$$

From (A.6) and using $\nabla a(r) = r^{-1} \boldsymbol{x} a'$ where $a(r)$ is a scalar function depending on the distance r one easily derives the following useful relations:

$$\frac{\partial}{\partial x_{\mu_{l+1}}} (a(r) X_{\mu_1 \dots \mu_l}) = \left(\frac{r}{2l+1} a' - a \right) X_{\mu_1 \dots \mu_{l+1}} + r^{-1} a' \frac{l(2l-1)}{2l+1} \Delta^{(l)}_{\mu_1 \dots \mu_l, \mu_{l+1} \nu_1 \dots \nu_{l-1}} X_{\nu_1 \dots \nu_{l-1}}, \quad \text{(A.12)}$$

$$\Delta (a(r) X_{\mu_1 \dots \mu_l}) = (a'' - 2l r^{-1} a') X_{\mu_1 \dots \mu_l} = r^{2l} (r^{-2l} a')' X_{\mu_1 \dots \mu_l}. \quad \text{(A.13)}$$

Here $\Delta = \partial^2 / \partial x_\varrho \partial x_\varrho$ is the Laplacian differential operator.

In view of Eq. (4.7) we note here that the multipole potentials are eigenfunctions of the differential operator $\mathfrak{L}$ (which is skin to the angular momentum operator of quantum mechanics)

$$\mathcal{L}_\mu = \varepsilon_{\mu\nu\lambda} x_\nu \frac{\partial}{\partial x_\lambda} = \varepsilon_{\mu\nu\lambda} \hat{x}_\nu \frac{\partial}{\partial \hat{x}_\lambda}. \quad \text{(A.14)}$$

Defining the isotropic $\Box^{(l)}$-tensor of rank $2l+1$ by

$$\Box^{(l)}_{\mu_1 \dots \mu_l, \lambda, \mu_1' \dots \mu'} = \Delta^{(l)}_{\mu_1 \dots \mu l, \nu_1 \dots \nu_{l-1} \nu_l} \varepsilon_{\nu_l \lambda \nu_l'} \Delta^{(l)}_{\nu_l' \nu_1 \dots \nu_{l-1}, \mu_1' \dots \mu_l'} = \frac{l+1}{l} \frac{2l+1}{2l+3} \Delta^{(l+1)}_{\mu_1 \dots \mu_l \mu, \mu' \mu_1' \dots \mu_l'} \varepsilon_{\mu\mu'\lambda}, \quad \text{(A.15)}$$

one has

$$\mathcal{L}_\lambda X_{\mu_1 \dots \mu_l} = l \Box^{(l)}_{\mu_1 \dots \mu_l, \lambda, \mu_1' \dots \mu_l'} X_{\mu_1' \dots \mu_l'}. \quad \text{(A.16)}$$

The normalization of the $\Box^{(l)}$-tensor is

$$\varepsilon_{\mu_l \lambda \mu_l'} \Box^{(l)}_{\mu_1 \dots \mu_{l-1} \mu_l, \lambda, \mu_1 \dots \mu_{l-1} \mu_l'} = \frac{l+1}{l} (2l+1). \quad \text{(A.17)}$$

The first two of these isotropic tensor are [10]

$$\Box^{(1)}_{\mu\lambda\mu'} = \varepsilon_{\mu\lambda\mu'}, \quad \Box^{(2)}_{\mu\nu, \lambda, \mu'\nu'} = \tfrac{1}{4} (\varepsilon_{\mu\lambda\mu'} \delta_{\nu\nu'} + \varepsilon_{\mu\lambda\nu'} \delta_{\nu\mu'} + \varepsilon_{\nu\lambda\mu'} \delta_{\mu\nu'} + \varepsilon_{\nu\lambda\nu'} \delta_{\mu\mu'}). \quad \text{(A.18)}$$

I thank Prof. Dr. L. WALDMANN for teaching me the application of Cartesian multipole potentials to problems in hydrodynamics and for many helpful comments.

Note added in proof: As Prof. D. A. KLEMM kindly pointed out to me, the result (5.6) for the transverse force has already been obtained by RUBINOW and KELLER [11].

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